A functional-analytic theory of vertex (operator) algebras, II

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Abstract

For a finitely-generated vertex operator algebra V of central charge $c \in \mathbb{C}$, a locally convex topological completion H^V is constructed. We construct on H^V a structure of an algebra over the operad of the $\frac{c}{2}$-th power $\mathrm{Det}^{c/2}$ of the determinant line bundle Det over the moduli space of genus-zero Riemann surfaces with ordered analytically parametrized boundary components. In particular, H^V is a module for the semi-group of the $\frac{c}{2}$-th power $\mathrm{Det}^{c/2}(1)$ of the determinant line bundle over the moduli space of conformal equivalence classes of annuli with analytically parametrized boundary components. The results in Part I for $\mathbb{Z}$-graded vertex algebras are also reformulated in terms of the framed little disk operad. Using May's recognition principle for double loop spaces, one immediate consequence of such operadic formulations is that the compactly generated spaces corresponding to (or the k-ifications of) the locally convex completions constructed in Part I and in the present paper have the weak homotopy types of double loop spaces. We also generalize the results above to locally-grading-restricted conformal vertex algebras and to modules.

0 Introduction

The present paper develops the functional-analytic aspects of vertex operator algebras. More specifically, we construct a locally convex topological completion of a finitely-generated vertex operator algebra and a structure on this completion of an algebra over a certain natural operad constructed from genus-zero Riemann surfaces with boundaries. We obtain representation-theoretic and homotopy-theoretic consequences and give generalizations to more general algebras and modules.

For a complex number c, consider the sequence $\mathrm{Det}^{c/2}$ of the $\frac{c}{2}$-th power of the determinant line bundles $\mathrm{Det}^{c/2}(n)$, $n \geq 0$, over the moduli spaces of genus-zero Riemann surfaces with $n+1$ ordered analytically parametrized boundaries. This sequence $\mathrm{Det}^{c/2}$ has a natural structure of (genuine) operad. (See [M1], [HL1], [HL2] and Appendix C of [H3] for the notion of operads and other related notions and see [Se1], [Se2] and Appendix D of [H3] for determinant line bundles.) An algebra over $\mathrm{Det}^{c/2}$ such that the underlying vector space is a complete locally convex topological vector space and the corresponding maps are continuous and depend holomorphic on $\mathrm{Det}^{c/2}$ is called a *genus-zero holomorphic conformal field theory of central charge c*. See [Se1] and [Se2] for a geometric definition of conformal field theory in the more general case of arbitrary genus and nonholomorphic theories.

Genus-zero conformal field theories are the starting point of a number of papers on algebraic structures derived from conformal field theories (see, for example, [KSV] [KVZ]). But the construction of examples of conformal field theories, even in this genus-zero case, is difficult and subtle. It has been expected that vertex operator algebras will give examples of such genus-zero holomorphic theories. But it is clear that vertex operator algebras themselves are not such theories. In fact, in [H1], [H2] and [H3], it was established that a vertex operator algebra has only the structure of an algebra over a $\mathbb{C}^{\times}$-rescalable partial operad in the sense of [HL1] and [HL2]. It is also clear that to construct such a theory from a vertex operator algebra, one first has to construct a suitable locally convex completion of the algebra. We know that $\mathrm{Det}^{c/2}$ is generated by $\mathrm{Det}^{c/2}(1)$ and $\mathrm{Det}^{c/2}(2)$, the $\frac{c}{2}$-th power of the determinant line bundle over of genus-zero Riemann surfaces with two and three, respectively, ordered analytically parametrized boundary components. Thus one must next construct continuous linear maps associated to elements in $\mathrm{Det}^{c/2}(1)$ and $\mathrm{Det}^{c/2}(2)$. Combining these maps with the geometric formulation of vertex operator algebras in terms of partial operads in [H3], it is easy to see that we will have a genus-zero holomorphic conformal field theory.

The main purpose of the present paper is to carry out this construction of genus-zero holomorphic conformal field theories from finitely-generated vertex operator algebras. The results in Part I for finitely-generated $\mathbb{Z}$-graded vertex algebras are also reformulated in terms of the framed little disk operad. Note that any genus-zero conformal field theory must be a representation of the semi-group $\mathrm{Det}^{c/2}(1)$, the $\frac{c}{2}$-th power of the determinant line bundle over the moduli space of annuli with analytically parametrized boundary

components. Thus, in particular, we construct in this paper a representation of $\mathrm{Det}^{c/2}(1)$ from a finitely-generated vertex operator algebra. In fact, from the construction it is easy to see that part of our construction actually gives a representation of $\mathrm{Det}^{c/2}(1)$ from an arbitrary $\mathbb{Z}$-graded representation of the Virasoro algebra satisfying a certain truncation condition. As far as the author knows, there seems to be no such general results on the integration of representations of the Virasoro algebra in the literature.

Combining the operadic formulations mentioned above with May's recognition principle for double loop spaces [M1], we conclude that the compactly generated spaces corresponding to (or the k-ifications of) the locally convex completions constructed in Part I and in the present paper have the weak homotopy types of double loop spaces. It is known that vertex operator algebras are a basic ingredient in conformal field theory and that conformal field theories describe string theory or M theory perturbatively. In string theory, there are two kinds of geometry involved, the "world-sheet" geometry and "space-time" geometry. The operad $\mathrm{Det}^{c/2}$ is part of the world-sheet geometry. The double loop space structures are interesting because they give us some "space-time" information about the vertex (operator) algebra. Since the operad $\mathrm{Det}^{c/2}$ has a much richer structure than the little disk operad, one should be able to recognize more properties of algebras over it. It will be much more interesting if one can recognize topological properties homeomorphically, not just (weak) homotopically, or even recognize some geometric properties. It will be especially interesting to see what geometric and topological properties can be recognized from structures associated to conformal field theories such as the minimal models which are constructed without any "space-time" geometry information.

These constructions and results above generalize to locally-grading-restricted conformal vertex algebras without any difficulty. These generalizations have been used in [HZ]. We also give the corresponding results for modules without giving detailed proofs.

The present paper is organized as follows: In Section 1, a locally convex topological completion H^V of a finitely-generated vertex operator algebra V of central charge $c \in \mathbb{C}$ is constructed. In Section 2, a structure of a representation on H^V of $\mathrm{Det}^{c/2}(1)$ is constructed. In Section 3, we construct linear continuous maps from the completed tensor product of two copies of H^V to H^V associated to elements of $\mathrm{Det}^{c/2}(2)$. In Section 4, we first reformulate the result in Part I ([H4]) in terms of the framed little disk operad. Then we state the main result (Theorem 4.2) of the present paper. Structures of

double loop spaces on the compactly generated spaces corresponding to (or the k-ifications of) the completions constructed in [H4] and in this paper are also stated in this section. The statements of the generalizations to locally-grading-restricted conformal vertex algebras and the corresponding results for modules are given in Sections 5 and 6, respectively.

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1 A locally convex completion of a finitely-generated vertex operator algebra

In this section, we construct a locally convex topological completion of a finitely-generated vertex operator algebra V. The topological completion is larger than the topological completion of a finitely-generated $\mathbb{Z}$-graded grading-restricted vertex algebra constructed in Part I ([H4]). For simplicity, we shall use the same notation H^V as in [H4] to denote the topological completion we shall construct in the present paper. But we warn the reader that H^V in the present paper is larger than H^V in [H4]. As in [H4], since V is fixed in the present paper, we shall denote H^V simply by H.

First we need to consider some geometric objects. A *disk* is a genus-zero Riemann surface with a connected boundary. A smooth invertible map from S^1 to the boundary of a disk is called an *analytic parametrization* if it can be extended to an analytic map from a neighborhood of S^1 inside the closed unit disk on the complex plane to a neighborhood of the boundary of the disk. A *disk with analytically parametrized boundary* is a disk equipped with an analytic parametrization of its boundary. For $k \geq 0$, a *k-punctured disk with analytically parametrized boundary* is a disk with analytically parametrized boundary and k ordered and distinct points in the interior of the disk. *Conformal equivalences* between k-punctured disks with analytically parametrized boundaries are defined in the obvious way.

Let $\Theta(k)$, $k \geq 0$, be the moduli spaces of k-punctured disks with analytically parametrized boundaries and let $\Theta = \cup_{k\geq 0}\Theta(k)$. Also consider the moduli spaces $\mathcal{B}_{0,1,k}$, $k \geq 0$, of genus-zero Riemann surfaces with ordered analytically parametrized boundary components, one positively oriented and

the other negatively oriented and ordered. The sequence $\{\mathcal{B}_{0,1,k}\}_{k\geq 0}$ has a natural structure of an analytic operad and this operad is isomorphic to the suboperad $K_{\mathfrak{H}_1}$ of the sphere partial operad K discussed in Section 6.4 of [H3]. It is clear that Θ has a natural structure of a space over the operad $\{\mathcal{B}_{0,1,k}\}_{k\geq 0}$ or equivalently over the operad $K_{\mathfrak{H}_1}$.

For any $k \geq 0$, we have an injective map from $\Theta(k)$ to $K(k)$ defined as follows: Take any element of $\Theta(k)$, that is, a conformal equivalence class of k-punctured disks with analytically parametrized boundaries. For any k-punctured disk with analytically parametrized boundary in this conformal equivalence class, by sewing the union of the exterior of S^1 and ∞ to this k-punctured disk using the analytic boundary parametrization, we obtain a $k+1$-punctured genus-zero Riemann surface, one puncture negatively oriented and the other puncture positively oriented and ordered, together with a local analytic coordinate vanishing at the negatively oriented puncture. Using the uniformization theorem, this $k+1$-punctured genus-zero Riemann surface with a local coordinate at the negatively oriented puncture is conformally equivalent to $\mathbb{C} \cup \{\infty\}$ with $k+1$ punctures, one negatively oriented and the other positively oriented and ordered, together with a local coordinate vanishing at the negatively oriented puncture. Moreover, we can choose the conformal equivalence (analytic diffeomorphism) such that the negatively oriented puncture is mapped to ∞, the k-th positively oriented puncture is mapped to 0 and the derivative at ∞ of the local coordinate map vanishing at ∞ is 1. Adding the standard local coordinates vanishing at the positively oriented punctures, we obtain a canonical sphere with tubes of type $(1, k)$ (see Chapter 3 of [H3]). It is clear that this canonical sphere with tubes of type $(1, k)$ is independent of the choice of the k-punctured disk with analytically parametrized boundary in the given conformal equivalence class. We define the image of the element of $\Theta(k)$ to be the conformal equivalence class of spheres with tubes of type $(1, k)$ containing this canonical sphere with tubes of type $(1, k)$. We obtain a map from $\Theta(k)$ to $K(k)$. Clearly this map is injective. We shall identify $\Theta(k)$ with its image in $K(k)$. Note that any element of $\Theta(k)$ viewed as an element of $K(k)$ is of the form

$$P = (z_1, \ldots, z_{k-1}; A, (1, \mathbf{0}), \ldots, (1, \mathbf{0})) \tag{1.1}$$

where $A \in H$. But also note that not all elements of $K(k)$ of this form is an element of $\Theta(k)$.

Note that $\Theta(k)$ can also be viewed as a subset of the Banach space $\mathbb{C}^{k-1} \times \mathrm{Hol}(D^1)$ where $\mathrm{Hol}(D^1)$ is the Banach space of all functions continuous on

the closed unit disk D^1 and holomorphic on the open unit disk. We give $\Theta(k)$ the topology and analytic structure induced from those on $\mathbb{C}^{k-1} \times \mathrm{Hol}(D^1)$.

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (in the sense of [FLM] and [FHL]). By the isomorphism theorem proved in Chapter 5 of [H3], there exists a canonical geometric vertex operator algebra structure on V. Let $\nu_k : K(k) \to \mathrm{Hom}(V^{\otimes k}, \overline{V})$, $k \geq 0$, be the maps defining the geometric vertex operator algebra structure on V. Then for any $v' \in V'$, any $u_1, \dots, u_k, v \in V$,

$$\langle v', (\nu_k(P))(u_1 \otimes \cdots \otimes u_k \otimes v)\rangle$$

as a function of P is meromorphic on $K(k)$. Thus for any $u_1, \dots, u_k, v \in V$ and any $P \in K(k)$, we have an element

$$Q(u_1, \dots, u_k, v; P) = (\nu_k(P))(u_1 \otimes \cdots \otimes u_k \otimes v) \in \overline{V}.$$

In particular, for any $u_1, \dots, u_k, v \in V$ and any $P \in \Theta(k)$, we have an element $Q(u_1, \dots, u_k, v; P) \in \overline{V}$ since $\Theta(k)$ can be viewed as a subset of $K(k)$.

For $k \geq 0$ and $n > 0$, let

$$\begin{aligned} J_n^{(k)} = & \{(z_1, \dots, z_{k-1}; A, (1, \mathbf{0}), \dots, (1, \mathbf{0})) \in \Theta(k) \mid |z_i - z_j| \geq \frac{1}{n}, i \neq j, \\ & |z_i| > \frac{1}{n}, i = 1, \dots, k, \text{the distances from } z_i,\ i = 1, \dots, k-1, \\ & 0 \text{ to } C_1 \text{ and from } 0 \text{ to } C_1^{-1} \text{ are large than or equal to } \frac{1}{n}\}, \end{aligned}$$

where

$$C_1 = f_A(\{w \in \mathbb{C} \mid |w| = 1\},$$
$$f_A(w) = e^{\sum_{j>0} A_j w^{j+1} \frac{d}{dw}} w$$

and

$$C_1^{-1} = \{w^{-1} \mid w \in C_1\}.$$

Then we see $\Theta(k) = \cup_{n>0} J_n^{(k)}, \quad k \geq 0$.

We denote the projections from V to $V_{(n)}$, $n \in \mathbb{Z}$, by P_n as in [H3]. For fixed $k \geq 0$, by the sewing axiom for geometric vertex operator algebras in [H3],

$$\sum_{n \in \mathbb{Z}} \langle v', (\nu_l(Q))(v_1 \otimes \cdots \otimes v_{l-1} \otimes (P_n(Q(u_1, \dots, u_k, v; P)))\rangle, \tag{1.2}$$

$v' \in V'$, $u_1, \dots, u_k, v_1, \dots, v_l \in V$, $P \in \Theta(k)$ and $Q \in K_{\mathfrak{H}_1}(l)$, is absolutely convergent. For fixed $v' \in V'$, $u_1, \dots, u_k, v_1, \dots, v_l \in V$, and $Q \in K_{\mathfrak{H}_1}(l)$, the sum of (1.2) give a function on Θ_k.

Lemma 1.1 *The functions defined by the sum of (1.2) is bounded on $J_n^{(k)}$, $n > 0$.*

Proof. By the sewing axiom for geometric vertex operator algebras in [H3], (1.2) is equal to

$$\langle v', (\nu_{k+l-1}(Q\,{}_l\infty_0\,P))(v_1 \otimes \cdots \otimes v_{l-1} \otimes u_1 \otimes \cdots \otimes u_k \otimes v)\rangle. \tag{1.3}$$

From (1.3) and the definition of ν_{k+l-1} in [H3], we see that to prove the lemma, we need only show that when $P \in J_n^{(k)}$, the distances between distinct punctures of $Q\,{}_l\infty_0\,P$ are larger than a fixed positive number depending only on n, and each expansion coefficient, as a function of P, of the analytic local coordinate maps vanishing at these punctures are bounded on $J_n^{(k)}$.

We first recall some facts and results from [H3]. Let

$$Q = (\xi_1, \dots, \xi_{l-1}; B^{(0)}, (b_0^{(1)}, B^{(1)}), \dots, (b_0^{(l)}, B^{(l)}))$$

and

$$f_{B^{(i)}, b_0^{(i)}}(w) = b_0^{(i)} e^{\sum_{j>0} B_j^{(i)} w^{j+1} \frac{d}{dw}} w$$

for $i = 0, \dots, l$. We shall also use the same notations $f_{B^{(i)}, b_0^{(i)}}$, $i = 0, \dots, l$, and f_A to denote the corresponding local coordinate maps. Then by the study of the sewing operation in [H3], the sewing equation

$$F^{(1)}(w) = F^{(2)}\left(f_A^{-1}\left(\frac{1}{f_{B^{(l)}, b_0^{(l)}}(w)}\right)\right)$$

together with the normalization conditions

$$\begin{aligned} F^{(1)}(\infty) &= \infty, \\ F^{(2)}(0) &= 0, \\ \lim_{w\to\infty} \frac{F^{(1)}}{w} &= 1 \end{aligned}$$

has a unique solution pair $F^{(1)}, F^{(2)}$ and the positively oriented punctures of $Q\,{}_l\infty_0\,P$ corresponding to the positively oriented punctures of P are $F^{(2)}(z_1)$, $\dots$, $F^{(2)}(z_{k-1})$ and 0. The local coordinate maps vanishing at these punctures are $F^{(2)}(w) - F^{(2)}(z_1), \dots, F^{(2)}(w) - F^{(2)}(z_l)$ and $F^{(2)}(w)$, respectively. It is also proved in [H3] that the sewing operation is analytic. In particular, it

is continuous. Thus $Q_l\infty_0 P$ is continuous in $P \in \Theta(k)$. In fact the proof actually proves that $F^{(1)}$ and $F^{(1)}$ depend on f_A and $f_{B^{(l)}, b_0^{(l)}}$ analytically and in particular continuously.

First we prove that when $P \in J_n^{(k)}$, the distances between distinct punctures of $Q_l\infty_0 P$ are larger than a fixed positive number depending only on n. If this is not true, then there is a sequence $\{P_m\}_{m>0}$ in $J_n^{(k)}$ and two punctures on $Q_l\infty_0 P_m$ for each $m > 0$ having the same orders, such that the distance between these two punctures goes to 0 when m goes to ∞. We consider the case that these two punctures are positively oriented punctures corresponding to two nonzero positively oriented punctures on P_m. If we use $z_1(P_m), \dots, z_{k-1}(P_m)$ to denote the punctures of P_m and $F_m^{(2)}$ and $F_m^{(2)}$ the solution of the sewing equation and the normalization conditions with P replaced by P_m, then by the results in [H3] we recalled above, these two punctures $Q_l\infty_0 P_m$ must be of the form $F_m^{(2)}(z_p(P_m))$ and $F_m^{(2)}(z_q(P_m))$ for some $0 < p, q < k$.

On the other hand, we can also obtain $z_1(P_m), \dots, z_{k-1}(P_m)$ from

$$F_m^{(2)}(z_1(P_m)), \dots, F_m^{(2)}(z_{k-1}(P_m))$$

as follows: We sew the first puncture of $(\mathbf{0}, (b_0^{(l)}, B^{(l)}(b_0^{(1)})))$ to the 0-the puncture of

$$(F_m^{(2)}(z_1(P_m)), \dots, F_m^{(2)}(z_{k-1}(P_m)); -\Psi^-, (1, \mathbf{0}), \dots, (1, \mathbf{0})),$$

where

$$B^{(l)}(b_0^{(l)}) = \{(b_0^{(l)})^j B_j^{(l)}\}_{j>0}$$

and $-\Psi^- = \{-\Psi_j\}_{j<0}$ is the sequence defined by

$$F^{(1)}(w) = e^{-\sum_{j<0} \Psi_j w^{j+1} \frac{d}{dw}} w.$$

Then the positively oriented punctures of the resulting element is $z_1(P_m), \dots,$ $z_k(P_m)$. In particular, $z_1(P_m), \dots, z_k(P_m)$ depend continuously on

$$F_m^{(2)}(z_1(P_m)), \dots, F_m^{(2)}(z_{k-1}(P_m)).$$

Thus since the distance between $F_m^{(2)}(z_p(P_m))$ and $F_m^{(2)}(z_q(P_m))$ goes to 0 when m goes to ∞, the distance between $z_p(P_m$ and $z_q(P_m$ must also goes to 0 when m goes to ∞. But $\{P_m\}_{m>0}$ is in $J_n^{(k)}$ and by the definition of

$J_n^{(k)}$, this is impossible. Similarly we get contradictions in the other cases. Thus when $P \in J_n^{(k)}$, the distances between distinct punctures of $Q_l\infty_0 P$ are larger than a fixed positive number depending only on n.

Now we prove that each expansion coefficient, as a function of P, of the analytic local coordinate maps vanishing at the punctures of $Q_l\infty_0 P$ is bounded on $J_n^{(k)}$. For simplicity, we prove this for the expansion coefficients of the analytic local coordinate map vanishing at the last puncture 0 of $Q_l\infty_0 P$. By the results in [H3] we recalled above, the local coordinate map vanishing at 0 is $(F^{(2)})^{-1}$. Since the expansion coefficients of $(F^{(2)})^{-1}$ at 0 are polynomials in the expansion coefficients of $F^{(2)}$, we need only show that each expansion coefficient of $F^{(2)}$ as a function of P is bounded on $J_n^{(k)}$. Note that the domain of $F^{(2)}$ contains C_1 and the interior of C_1. When $P \in J_n^{(k)}$, the union of C_1 and the interior of C_1 always contains the closed disk centered at 0 of radius $1/n$. Since the radius $1/n$ is independent of P, using the Cauchy formulas for the expansion coefficients of $F^{(2)}$, we see that each of these coefficients is bounded on $J_n^{(k)}$. ∎

Let $\tilde{G}$ be the subspace of V^* consisting of linear functionals λ on V such that for any $k \geq 0$, $u_1, \ldots, u_k, v \in V$, $P \in \Theta(k)$,

$$\sum_{n\in\mathbb{Z}} \lambda(P_n(Q(u_1, \ldots, u_k, v; P))) \tag{1.4}$$

is absolutely convergent and its sum as a function on $\Theta(k)$ is bounded on $J_n^{(k)}$, $n > 0$. The dual pair (V^*, V) of vector spaces gives V^* a locally convex topology. With the topology induced from the one on V^*, $\tilde{G}$ is also a locally convex space. Note that V' is a subspace of $\tilde{G}$. (We warn the reader that $\tilde{G}$ here is different from $\tilde{G}$ in [H4]. In the present paper, many notations we use are the same as the corresponding notations in [H4]. But what they denote are different from what the same notations denote in [H4].)

We denote the analytic function on $\Theta(k)$ defined by (1.4) by

$$g_k(\lambda \otimes u_1 \otimes \cdots \otimes u_k \otimes v)$$

since (1.4) is multilinear in λ, $u_1, \ldots, u_k$ and v. These functions span a vector space F_k of analytic functions on $\Theta(k)$. We obtain a linear map

$$g_k : \tilde{G} \otimes V^{\otimes(k+1)} \to F_k.$$

By definition, elements of F_k are bounded on $J_n^{(k)}$, $n > 0$.

We define a family of norms $\|\cdot\|_{F_k,n}$, $n > 0$, on F_k by

$$\|g\|_{F_k,n} = \sup_{Q\in J_n^{(k)}} g(Q)$$

for $g \in F_k$. These norms give a locally convex topology on F_k. Note that a net $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ (where $\mathcal{A}$ is an index set) in F_k is convergent to $f \in \Theta_k$ if and only if it is convergent uniformly in $J_n^{(k)}$ for $n > n_0$ where n_0 is a positive integer.

For any $k \geq 0$, there is an embedding ι_{F_k} from F_k to F_{k+1} defined as follows: We use

$$(z_0, \dots, z_{k-1}; A, (1, \mathbf{0}), \dots, (1, \mathbf{0}))$$

instead of

$$(z_1, \dots, z_k; A, (1, \mathbf{0}), \dots, (1, \mathbf{0}))$$

to denote the elements of Θ_{k+1}. For $\lambda \in \tilde{G}$, $u_1, \dots, u_k, v \in V$, since

$$Y(\mathbf{1}, z) = 1$$

for any nonzero complex number z,

$$g_{k+1}(\lambda \otimes \mathbf{1} \otimes u_1 \otimes \cdots \otimes u_k \otimes v)$$

as a function of $(z_0, \dots, z_{k-1}; A, (1, \mathbf{0}), \dots, (1, \mathbf{0}))$ is in fact independent of z_0, and is equal to

$$g_k(\lambda \otimes u_1 \otimes \cdots \otimes u_k \otimes v)$$

as a function in $(z_1, \dots, z_{k-1}; A, (1, \mathbf{0}), \dots, (1, \mathbf{0}))$. Thus we obtain a well-defined linear map

$$\iota_{F_k} : F_k \to F_{k+1}$$

such that

$$\iota_{F_k} \circ g_k = g_{k+1} \circ \phi_k$$

where

$$\phi_k : \tilde{G} \otimes V^{\otimes(k+1)} \to \tilde{G} \otimes V^{\otimes(k+2)}$$

is defined by

$$\phi_k(\lambda \otimes u_1 \otimes \cdots \otimes u_k \otimes v) = \lambda \otimes \mathbf{1} \otimes u_1 \otimes \cdots \otimes u_k \otimes v$$

for $\lambda \in \tilde{G}$, $u_1, \dots, u_k, v \in V$. It is clear that ι_{F_k} is injective. Thus we can regard F_k as a subspace of F_{k+1}. Moreover, we have:

Proposition 1.2 *For any $k \geq 0$, ι_{F_k} as a map from F_k to $\iota_{F_k}(F_k)$ is continuous and open. In other words, the topology on F_k is induced from that on F_{k+1}.*

Proof. We consider the two topologies on F_k, one is the topology defined above for F_k and the other induced from the topology on F_{k+1}. We need only prove that for any $n > 0$, (i) the norm $\|\cdot\|_{F_k,n}$ is continuous in the topology induced from the one on F_{k+1}, and (ii) the restriction of the norm $\|\cdot\|_{F_{k+1},n}$ to F_k is continuous in the topology on F_k.

Let $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ (where $\mathcal{A}$ is an index set) be a net in F_k convergent in the topology induced from the one on F_{k+1}. Then $\{f_\alpha\}_{\alpha\in\mathcal{A}}$, when viewed as a net of functions in

$$(z_0, z_1, \dots, z_{k-1}; A, (\mathbf{0}, (1, \mathbf{0}), \dots, (\mathbf{0}, (1, \mathbf{0}))$$

is convergent uniformly on $J_n^{(k+1)}$, $n > 0$. Since f_α, $\alpha \in \mathcal{A}$, are independent of z_0, $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is in fact convergent uniformly on the sets $J_{n+1}^{(k)}$, $n > 0$, proving (i). Now let $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ be a net in F_k convergent in the topology on F_k. Then $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is convergent uniformly on $J_n^{(k)}$, $n > 0$. If we view f_α, $\alpha \in \mathcal{A}$, as functions on $\mathbb{C} \times \Theta(k)$, then the net $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is convergent uniformly on $(\mathbb{C} \times J_n^{(k)}) \cap \Theta(k+1)$ (where we view $\Theta(k+1)$ as a subset of $\mathbb{C} \times \Theta(k)$. Since $J_n^{(k+1)} \subset (\mathbb{C} \times J_n^{(k)}) \cap \Theta(k+1)$, $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is convergent uniformly on $J_n^{(k+1)}$, $n > 0$, proving (ii). ∎

We equip the topological dual space F_k^*, $k \geq 0$, of F_k with the strong topology, that is, the topology of uniform convergence on all the weakly bounded subsets of F_k. Then F_k^* is a locally convex space.

For $k \geq 0$, we define a linear map

$$\gamma_k : F_{k+1} \to F_k$$

as follows: We use $P = (z_0, \dots, z_{k-1}; A, (1, \mathbf{0}), \dots, (1, \mathbf{0}))$ to denote an element of Θ_{k+1}. Recall that

$$C_1 = f_A(\{w \in \mathbb{C} \mid |w| = 1\}$$

and

$$f_A(w) = e^{\sum_{j>0} A_j w^{j+1} \frac{d}{dw}} w.$$

We define

$$\begin{aligned}
&\gamma_k(g_{k+1}(\lambda\otimes u_0\otimes u_1\otimes\cdots\otimes u_k\otimes v))\\
&\quad=\frac{1}{2\pi\sqrt{-1}}\oint_{C_1} z_0^{-1}g_{k+1}(\lambda\otimes u_0\otimes u_1\otimes\cdots\otimes u_k\otimes v)dz_0
\end{aligned}\tag{1.5}$$

for $\lambda\in\tilde{G}$, $u_0,u_1,\dots,u_k,v\in V$.

We still need to show that the right-hand side of (1.5) is indeed in F_k. Let $P'=(z_1,\dots,z_{k-1};A,(1,\mathbf{0}),\dots,(1,\mathbf{0}))\in\Theta_k$. Then we have

$$P=(f_A^{-1}(z_0);\mathbf{0},B(z_0),(1,\mathbf{0}))_2\infty_0 P' \tag{1.6}$$

(see formula (A.6.1) in [H3]), where

$$B(z_0)=\hat{E}^{-1}\left(\frac{1}{f_A^{-1}\left(\frac{1}{x+\frac{1}{f_A(z_0)}}\right)}-\frac{1}{z_0}\right).$$

By the definition of g_k and (1.6), we have

$$\begin{aligned}
&\gamma_k(g_{k+1}(\lambda\otimes u_0\otimes u_1\otimes\cdots\otimes u_k\otimes v))\\
&\quad=\frac{1}{2\pi\sqrt{-1}}\oint_{C_1} z_0^{-1}\sum_{n\in\mathbb{Z}}\lambda(P_n(Q(u_0,\dots,u_k,v;P)))dz_0.
\end{aligned}\tag{1.7}$$

Since the series

$$\sum_{n\in\mathbb{Z}}\lambda(P_n(Q(u_0,\dots,u_k,v;P)))$$

is absolutely convergent, the right-hand side of (1.7) is equal to

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\sum_{n\in\mathbb{Z}}\oint_{C_1} z_0^{-1}\lambda(P_n(\nu_{k+1}(P)(u_0\otimes u_1\otimes\cdots\otimes u_k\otimes v)))dz_0\\
&\quad=\frac{1}{2\pi\sqrt{-1}}\sum_{n\in\mathbb{Z}}\oint_{C_1} z_0^{-1}\lambda(P_n((\nu_{k+1}((f_A^{-1}(z_0);\mathbf{0},B(z_0),(1,\mathbf{0}))_2\infty_0 P')\\
&\qquad\qquad(u_0\otimes u_1\otimes\cdots\otimes u_k\otimes v)))dz_0\\
&\quad=\frac{1}{2\pi\sqrt{-1}}\sum_{n\in\mathbb{Z}}\oint_{C_1} z_0^{-1}\lambda(P_n((\nu_2((f_A^{-1}(z_0);\mathbf{0},B(z_0),(1,\mathbf{0})))_2*_0\nu_k(P'))\\
&\qquad\qquad(u_0\otimes u_1\otimes\cdots\otimes u_k\otimes v)))dz_0
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi\sqrt{-1}} \sum_{n\in\mathbb{Z}} \oint_{C_1} z_0^{-1}\lambda(Y(e^{-\sum_{j>0} B(z_0)L(j)}u_0, f_A^{-1}(z_0))\\
&\qquad\qquad P_n((\nu_k(P'))(u_1\otimes\cdots\otimes u_k\otimes v)))dz_0\\
&= \frac{1}{2\pi\sqrt{-1}} \sum_{n\in\mathbb{Z}} \oint_{|w|=1} (f_A(w))^{-1}f_A'(w)\lambda(Y(e^{-\sum_{j>0} B(f_A(w))L(j)}u_0, w)\\
&\qquad\qquad P_n((\nu_k(P'))(u_1\otimes\cdots\otimes u_k\otimes v)))dw\\
&= \sum_{n\in\mathbb{Z}} \mathrm{Res}_w (f_A(w))^{-1}f_A'(w)\lambda(Y(e^{-\sum_{j>0} B(f_A(w))L(j)}u_0, w)\\
&\qquad\qquad P_n(\nu_k(P'))(u_1\otimes\cdots\otimes u_k\otimes v))).
\end{aligned}
\tag{1.8}
$$

Let $\tilde{\lambda}$ be an element of V' defined by

$$
\tilde{\lambda}(v) = \mathrm{Res}_w (f_A(w))^{-1}f_A'(w)\lambda(Y(e^{-\sum_{j>0} B(f_A(w))L(j)}u_0, w)v).
$$

Then by (1.7) and (1.8), $\tilde{\lambda}\in\tilde{G}$ and thus the right-hand side of (1.5) is in F_k.

Proposition 1.3 *The map γ_k is continuous and satisfies*

$$
\gamma_k\circ\iota_{F_k} = I_{F_k} \tag{1.9}
$$

where I_{F_k} is the identity map on F_k.

Proof. We still use

$$
(z_0,\dots,z_{k-1}; A, (1,\mathbf{0}),\dots,(1,\mathbf{0}))
$$

instead of

$$
(z_1,\dots,z_k; A, (1,\mathbf{0}),\dots,(1,\mathbf{0}))
$$

to denote an element of Θ_{k+1}. We know that there exists $t\in[0,1)$ such that for $\epsilon\in[t,1]$, $\{w\in\mathbb{C}\mid |w|=\epsilon\}$ is in the domain of $f_A(1/w)$. Let $C_\epsilon = f_A(\{w\in\mathbb{C}\mid |w|=1/\epsilon\})$ for $\epsilon\in[t,1]$. Then by the definition of γ_k and Cauchy's theorem, for any $\epsilon\in[t,1]$ such that $z_1,\dots,z_{k-1}$ are in the interior of C_ϵ, we have

$$
\begin{aligned}
&\gamma_k(g_{k+1}(\lambda\otimes u_0\otimes u_1\otimes\cdots\otimes u_l\otimes v))\\
&\qquad = \frac{1}{2\pi\sqrt{-1}}\oint_{C_\epsilon} z_0^{-1} g_{k+1}(\lambda\otimes u_0\otimes u_1\otimes\cdots\otimes u_k\otimes v)dz_0
\end{aligned}
$$

for $\lambda \in \tilde{G}$, $u_0, \dots, u_k, v \in V$. Thus by the definition of $J_n^{(k)}$, for any $n > 0$, there exists $\epsilon_n \in [t, 1]$ such that

$$\begin{aligned}
&\|\gamma_k(g_{k+1}(\lambda \otimes u_0 \otimes u_1 \otimes \cdots \otimes u_k \otimes v))\|_{F_k,n} \\
&\quad= \sup_{(z_1,\dots,z_{k-1};A,(1,\mathbf{0}),\dots,(1,\mathbf{0}))\in J_n^{(k)}} |\gamma_k(g_{k+1}(\lambda \otimes u_0 \otimes u_1 \otimes \cdots \otimes u_k \otimes v))| \\
&\quad= \sup_{(z_1,\dots,z_{k-1};A,(1,\mathbf{0}),\dots,(1,\mathbf{0}))\in J_n^{(k)}} \\
&\qquad\qquad \Big|\frac{1}{2\pi\sqrt{-1}}\oint_{z_0\in C_{\epsilon_n}} z_0^{-1} g_{k+1}(\lambda \otimes u_0 \otimes u_1 \otimes \cdots \otimes u_k \otimes v) dz_0\Big| \\
&\quad\le \sup_{(z_1,\dots,z_{k-1};A,(1,\mathbf{0}),\dots,(1,\mathbf{0}))\in J_n^{(k)}, z_0\in C_{\epsilon_n}} |g_{k+1}(\lambda \otimes u_0 \otimes u_1 \otimes \cdots \otimes u_k \otimes v)|.
\end{aligned} \tag{1.10}$$

For any $z_0 \in C_{\epsilon_n}$, it is clear that there always exists positive integer n_{z_0} and a open subset U_{z_0} of $\mathbb{C}$ containing z_0 such that $U_{z_0} \times J_n^{(k)} \subset J_{n_{z_0}}^{(k+1)}$. Since C_{ϵ_n} is compact, there exists finitely many points $z_0^{(1)}, \dots, z_0^{(l)} \in C_{\epsilon_n}$ such that $U_{z_0^{(1)}}, \dots, U_{z_0^{(l)}}$ cover C_{ϵ_n}. Thus the right-hand side of (1.10) is less than or equal to

$$\begin{aligned}
&\sum_{i=1}^{l} \max_{(z_0,\dots,z_{k-1};A,(1,\mathbf{0}),\dots,(1,\mathbf{0}))\in J_{n_{z_0^{(i)}}}^{(k+1)}} |g_{k+1}(\lambda \otimes u_0 \otimes u_1 \otimes \cdots \otimes u_k \otimes v)| \\
&\qquad = \sum_{i=1}^{l} \|g_{k+1}(\lambda \otimes u_0 \otimes u_1 \otimes \cdots \otimes u_k \otimes v)\|_{F_{k+1}, n_{z_0^{(i)}}}.
\end{aligned} \tag{1.11}$$

Combining (1.10) and (1.11), we see that γ_k is continuous.

For $\lambda \in \tilde{G}$, $u_1, \cdots, u_k, v \in V$, by definition,

$$\begin{aligned}
&g_{k+1}(\lambda \otimes \mathbf{1} \otimes u_1 \otimes \cdots \otimes u_k \otimes v) \\
&\qquad = \iota_{F_k}(g_k(\lambda \otimes u_1 \otimes \cdots \otimes u_k \otimes v)).
\end{aligned}$$

By definition,

$$g_{k+1}(\lambda \otimes \mathbf{1} \otimes u_1 \otimes \cdots \otimes u_k \otimes v) = g_k(\lambda \otimes u_1 \otimes \cdots \otimes u_k \otimes v).$$

Thus

$$\begin{aligned}
&\gamma_k(g_{k+1}(\lambda \otimes \mathbf{1} \otimes u_1 \otimes \cdots \otimes u_k \otimes v)) \\
&\qquad = g_k(\lambda \otimes u_1 \otimes \cdots \otimes u_k \otimes v).
\end{aligned}$$

So we have (1.9). ∎

The proof of the following consequence is the same as the proof of Corollary 1.3 in [H4]:

Corollary 1.4 *The adjoint map γ_k^* of γ_k satisfies*

$$\iota_{F_k}^* \circ \gamma_k^* = I_{F_k^*} \tag{1.12}$$

where

$$\iota_{F_k}^* : F_{k+1}^* \to F_k^*$$

is the adjoint of ι_{F_k} and $I_{F_k^*}$ is the identity on F_k^*. It is injective and continuous. As a map from F_k^* to $\gamma_k^*(F_k^*)$, it is also open. In particular, if we identify F_k^* with $\gamma_k^*(F_k^*)$, the topology on F_k^* is induced from the one on F_{k+1}^*. ∎

In the rest of this section, we give the remaining steps in the construction of the locally convex completion. These steps are mostly the same as those in [H4]. Thus our description of these steps shall be brief. Also we warn the reader again that although the notations we use below are the same as those in [H4], they denote different things in the present paper.

We use $\langle \cdot, \cdot \rangle$ to denote the pairing between $\tilde{G}$ and the algebraic dual space $\tilde{G}^*$ of $\tilde{G}$. It is an extension of the pairing between V' and V denoted using the same symbol. The spaces $\tilde{G}$ and $\tilde{G}^*$ with this pairing form a dual pair of vector spaces and thus give a locally convex topology to $\tilde{G}^*$. The dual space $\tilde{G}^*$ can be viewed as a subspace of $(V')^* = \overline{V}$. We define

$$e_k : V^{\otimes (k+1)} \otimes F_k^* \to \tilde{G}^* \subset \overline{V}$$

by

$$\langle \lambda, e_k(u_1 \otimes \cdots \otimes u_k \otimes v \otimes \mu) \rangle = \mu(g_k(\lambda \otimes u_1 \otimes \cdots \otimes u_k \otimes v))$$

for $\lambda \in \tilde{G}$, $u_1, \dots, u_k, v \in V$ and $\mu \in F_k^*$.

We now have to assume that V is finitely generated. Let X be the finite-dimensional subspace X of V spanned by a finite set of generators of V containing the vacuum vector $\mathbf{1}$. We give X the topology induced by any norm on X. Then $X^{\otimes (k+1)} \otimes F_k^*$ is a locally convex space. Let G_k be the image $e_k(X^{\otimes (k+1)} \otimes F_k^*)$ of $X^{\otimes (k+1)} \otimes F_k^* \subset V^{\otimes (k+1)} \otimes F_k^*$ under e_k.

The proofs of Propositions 1.5 and 1.6 below are the same as the proofs of Propositions 1.4 and 1.5 in [H4]:

Proposition 1.5 *For any $k \geq 0$, $G_k \subset G_{k+1}$.* ∎

Proposition 1.6 *The linear map*

$$e_k|_{X^{\otimes(k+1)}\otimes F_k^*} : X^{\otimes(k+1)} \otimes F_k^* \to \tilde{G}^*$$

is continuous. ∎

Corollary 1.7 *The quotient space*

$$(X^{\otimes(k+1)} \otimes F_k^*)/(e_k|_{X^{\otimes(k+1)}\otimes F_k^*})^{-1}(0)$$

is a locally convex space. ∎

Using the isomorphism from G_k to

$$(X^{\otimes(k+1)} \otimes F_k^*)/(e_k|_{X^{\otimes(k+1)}\otimes F_k^*})^{-1}(0),$$

we obtain a locally convex space structure on G_k from that on

$$(X^{\otimes(k+1)} \otimes F_k^*)/(e_k|_{X^{\otimes(k+1)}\otimes F_k^*})^{-1}(0).$$

Let H_k be the completion of G_k. Then H_k is a complete locally convex space.

The proof of the following proposition is the same as the proof of Proposition 1.7 in [H4]:

Proposition 1.8 *The space H_k can be embedded canonically in H_{k+1}. The topology on H_k is the same as the one induced from the topology on H_{k+1}.* ∎

Now we have a sequence $\{H_k\}_{k\geq 0}$ of strictly increasing complete locally convex spaces. Let

$$H = \bigcup_{k\geq 0} H_k$$

equipped with the inductive limit topology. Then H is a complete locally convex space. Let

$$G = \bigcup_{k\geq 0} G_k \subset H.$$

Then $V \subset G$ and G is dense in H. The same argument as in [H4] shows that G is in the closure of V. Thus we have:

Theorem 1.9 *The vector space H equipped with the strict inductive limit topology is a locally convex completion of V.* ∎

2 The locally convex completion and a semi-group of annuli

In this section, we construct, on the topological completion H, a structure of a representation of the semi-group of the $\frac{c}{2}$-th power of the determinant line bundle over the moduli space of conformal equivalence classes of annuli with analytically parametrized boundary components.

Consider the moduli space $\mathcal{B}_{1,1,0}$ of annuli, that is, the space of conformal equivalence classes of genus-zero Riemann surfaces with two boundary components, one positively oriented and one negatively oriented, and with analytic boundary parametrizations of the boundary components. There is a sewing operation on $\mathcal{B}_{1,1,0}$ such that it becomes a semi-group. (See Appendix D of [H3] for details.) There is a determinant line bundle $\mathrm{Det}(1)$ over $\mathcal{B}_{1,1,0}$ and its c-th power $\mathrm{Det}^c(1)$ for any $c \in \mathbb{C}$ is well-defined.

Proposition 2.1 *For any complex number c, $\mathrm{Det}^c(1)$ has a structure of a semi-group and is the central extension of $\mathcal{B}_{1,1,0}$ with central charge $2c$.* ∎

This result and its proof are contained implicitly in Appendix D of [H3]. See [H3] for details.

By the uniformization theorem, it is clear that the semi-group $\mathcal{B}_{1,1,0}$ is isomorphic to the semi-group of the moduli space $K_{\mathfrak{H}_1}(1)$ equipped with the sewing operation. We shall identify $\mathcal{B}_{1,1,0}$ with $K_{\mathfrak{H}_1}(1)$. Over the moduli space $K(1)$, we have a determinant line bundle and its $\frac{c}{2}$-th power $\tilde{K}^c(1)$ for any complex number c. We denote the restriction of $\tilde{K}^c(1)$ to $K_{\mathfrak{H}_1}(1)$ by $\tilde{K}^c_{\mathfrak{H}_1}(1)$. Then $\tilde{K}^c_{\mathfrak{H}_1}(1)$ is a semi-group isomorphic to $\mathrm{Det}^{c/2}(1)$. See [H3] for details. We now construct a structure of a representation of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on H where c is the central charge of V.

First we give a right action of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on $\tilde{G}$. Let $\lambda \in \tilde{G}$ and $\tilde{Q} = (Q; C) \in \tilde{K}^c_{\mathfrak{H}_1}(1)$ (where $Q \in K_{\mathfrak{H}_1}(1)$ and $C \in \mathbb{C}$). We define $\lambda_{\tilde{Q}} \in V^*$ by

$$\lambda_{\tilde{Q}}(v) = C \sum_{n \in \mathbb{Z}} \lambda(P_n((\nu_1(Q))(v)). \tag{2.1}$$

Note that the right-hand side of (2.1) is absolutely convergent because $\lambda \in \tilde{G}$.

Lemma 2.2 *The linear functional $\lambda_{\tilde{Q}}$ is in fact in $\tilde{G}$.*

Proof. By definition, for any $P \in \Theta(k)$,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \lambda_{\tilde{Q}}(P_n(Q(u_1,\dots,u_k,v;P)))\\
&\quad = \sum_{n\in\mathbb{Z}} \lambda_{\tilde{Q}}(P_n((\nu_k(P))(u_1\otimes\cdots\otimes u_k\otimes v)))\\
&\quad = C\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \lambda(P_m((\nu_1(Q))(P_n((\nu_k(P))(u_1\otimes\cdots\otimes u_k\otimes v))))). \qquad (2.2)
\end{aligned}$$

We want to show that the right-hand side of (2.2) is absolutely convergent. To show this convergence, we note that, by the sewing axiom for geometric vertex operator algebras,

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \lambda(P_m((\nu_1(Q))(P_n((\nu_k(P))(u_1\otimes\cdots\otimes u_k\otimes v)))))\\
&\quad = \sum_{m\in\mathbb{Z}} \lambda(P_m((((\nu_1(Q))_1 *_0 (\nu_k(P)))(u_1\otimes\cdots\otimes u_k\otimes v))))\\
&\quad = \sum_{m\in\mathbb{Z}} \lambda(P_m((\nu_k(Q_1\infty_0 P))(u_1\otimes\cdots\otimes u_k\otimes v))). \qquad (2.3)
\end{aligned}$$

Note that since $\lambda\in\tilde{G}$, the right-hand side of (2.3) is absolutely convergent and is analytic in P and Q. Thus the double sum and the iterated sum in the other order are also absolutely convergent. Since the iterated sum in the right-hand side of (2.2) is exactly the iterated sum in the other order, it is absolutely convergent. ∎

By this lemma, $\lambda\mapsto\lambda_{\tilde{Q}}$ for $\lambda\in\tilde{G}$ give a right action of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on $\tilde{G}$. This right action induces a left action on $\tilde{G}^*$. It also induces right actions of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on F_k, $k\geq 0$, as follows:

$$\begin{aligned}
g_k(\lambda\otimes u_1\otimes\cdots\otimes u_k\otimes v) \;\mapsto\; & g_k^{\tilde{Q}}(\lambda\otimes u_1\otimes\cdots\otimes u_k\otimes v)\\
&= g_k(\lambda_{\tilde{Q}}\otimes u_1\otimes\cdots\otimes u_k\otimes v),
\end{aligned}$$

for $\tilde{Q}\in\tilde{K}^c_{\mathfrak{H}_1}(1)$, $\lambda\in\tilde{G}$, $u_1,\dots,u_k,v\in V$. These right actions on F_k, $k\geq 0$, induce left actions on F_k^*. For simplicity, we shall also use $\tilde{Q}$ to denote the images of $\tilde{Q}\in\tilde{K}^c_{\mathfrak{H}_1}(1)$ in End $\tilde{G}^*$ and End F_k^*, $k\geq 0$.

Proposition 2.3 *For $k\geq 0$, $\tilde{Q}\in\tilde{K}^c_{\mathfrak{H}_1}(1)$, $\mu\in F_k^*$, $u_1,\dots,u_k,v\in V$.*

$$\tilde{Q}\cdot e_k(u_1\otimes\cdots\otimes u_k\otimes v\otimes\mu) = e_k(u_1\otimes\cdots\otimes u_k\otimes v\otimes\tilde{Q}\cdot\mu).$$

Proof. This follows from the definitions of e_k and the left actions of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on $\tilde{G}^*$ and F_k^*. ∎

By this proposition, we immediately obtain:

Corollary 2.4 *For $k \geq 0$, the actions of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on $\tilde{G}^*$ and F_k^* induce an action of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on G_k and thus an action on H_k. The actions of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on H_k induce an action on H.* ∎

We shall still use $\tilde{Q}$ to denote the images of $\tilde{Q} \in \tilde{K}^c_{\mathfrak{H}_1}(1)$ in End H_k, $k \geq 0$, and End H. We have the following:

Proposition 2.5 *Let $\tilde{P} \in \tilde{K}^c_{\mathfrak{H}_1}(1)$. Then its images in* End H_k*, $k \geq 0$, and* End H *are continuous.*

Proof. We need only prove the continuity of the images of $\tilde{Q}$ in End H_k, $k \geq 0$. Since the actions on H_k, $k \geq 0$, are induced from the action on $\tilde{G}^*$, we need only show that the image of $\tilde{Q}$ in End $\tilde{G}^*$ is continuous. This is equivalent to the continuity of the image of $\tilde{Q}$ in End $\tilde{G}$. But from the definition (2.1), it is clear that the image of $\tilde{Q}$ in End $\tilde{G}$ is continuous. ∎

Combining Corollary 2.4 and Proposition 2.5, we obtain the following:

Theorem 2.6 *The complete locally convex spaces H_k, $k \geq 0$, and H have structures of continuous representations of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ or of* $\mathrm{Det}^{c/2}(1)$*.* ∎

Note that in the constructions of H_0 and of the structure of a continuous representation of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ on H_0, only the structure of a $\mathbb{Z}$-graded representation of the Virasoro algebra on V and a certain lower-truncation condition of the representation is used. Thus we actually have the following:

Theorem 2.7 *Let $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ be a $\mathbb{Z}$-graded module for the Virasoro algebra satisfying the conditions:* (i) *$L(0)v = nv$ for $v \in V_{(n)}$ and* (ii) *for any $v \in V$, the $\mathbb{Z}$-graded submodule $W = \coprod_{n\in\mathbb{Z}} W_{(n)}$ for the Virasoro algebra generated by v is lower truncated, that is, $W_{(n)} = 0$ when n is sufficiently small. Then the same constructions in Section 1 and in this section gives a locally convex completion H_0 of V and a structure of a continuous representation of $\tilde{K}^c_{\mathfrak{H}_1}(1)$ or of* $\mathrm{Det}^{c/2}(1)$ *on H_0.* ∎

3 The locally convex completion and the vertex operator map

Consider a conformal equivalence class of genus-zero Riemann surfaces with three ordered boundary components, the first positively oriented and the other two negatively oriented, and with analytic parametrizations at these boundary components. Such a conformal equivalence class can be naturally identified with an element of $K_{\mathfrak{H}_1}(2)$ (see [H3]). We shall denote the corresponding element in $K_{\mathfrak{H}_1}(2)$ by Q. Then a pair consisting of such a conformal equivalence class and an element of the $\frac{c}{2}$-th power of the determinant line over it corresponding to an element $\tilde{Q}$ of $\tilde{K}^c_{\mathfrak{H}_1}(2)$.

In this section, we use the vertex operator map to construct continuous linear maps from the topological completion of $H \otimes H$ to H associated to $\tilde{Q} \in \tilde{K}^c_{\mathfrak{H}_1}(2)$.

Let $H \tilde{\otimes} H$ be the locally convex completion of the vector space tensor product $H \otimes H$. We would like to construct a continuous linear map

$$\overline{\Psi}_Y(\tilde{Q}) : H \tilde{\otimes} H \to H$$

associated to $\tilde{P}$ such that restricting to $V \otimes V$, it is equal to the linear map $\Psi_Y(\tilde{Q}) : V \otimes V \to \overline{V}$ constructed in [H3]. Because $\tilde{K}^c_{\mathfrak{H}_1}(2)$ is infinite-dimensional, our construction here is more complicated than the one in [H4]. Nevertheless, the idea and the steps are mostly the same. Because of this, we shall be brief in our arguments below.

Given any $Q \in K(2)$, let Q' be the element of $K(2)$ obtained by switching the negatively oriented and the second positively oriented punctures of any sphere with tubes in P. Thus we obtain a bijective map $'$ from $K(2)$ to itself. Since the line bundle $\tilde{K}^c(2)$ is canonically trivial, this map $'$ can be extended to a bijective map $'$ from $\tilde{K}^c(2)$ to itself. It is clear that this map $'$ map $\tilde{K}^c_{\mathfrak{H}_1}(2)$ to itself.

We now fix $\tilde{Q} \in \tilde{K}^c_{\mathfrak{H}_1}(2)$. For any $\lambda \in \tilde{G}$ and $u \in V$, we define an element $u \diamond_{\tilde{Q}} \lambda \in V^*$ by

$$(u \diamond_{\tilde{Q}} \lambda)(v) = \sum_{n \in \mathbb{Z}} \lambda(P_n((\Psi_2(\tilde{Q}'))(u \otimes v))$$

for $v \in V$.

Proposition 3.1 *The element $u \diamond_{\tilde{Q}} \lambda$ is in $\tilde{G}$.*

Proof. We write $\tilde{Q} = (Q; C)$. For any $k \geq 0$, $u_1, \ldots, u_k, v \in V$, $P \in \Theta_k$,

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} (u \diamond_{\tilde{Q}} \lambda)(P_m(Q(u_1, \ldots, u_k, v; P)))\\
&\quad = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \lambda(P_n((\Psi_2(\tilde{Q}'))(u \otimes P_m(Q(u_1, \ldots, u_k, v; P))))\\
&\quad = C \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \lambda(P_n((\nu_2(Q'))(u \otimes P_m(Q(u_1, \ldots, u_k, v; P))))\\
&\quad = C \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \lambda(P_n((\nu_2(Q'))(u \otimes P_m(Q(u_1, \ldots, u_k, v; P))))\\
&\quad = C \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \lambda(P_n(Q(u, P_m(Q(u_1, \ldots, u_k, v; P)); Q'))). \qquad (3.1)
\end{aligned}$$

We need to prove that the right-hand side of (3.1) is absolutely convergent.

As in [H4], we consider the iterated sum in the other order

$$C \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \lambda(P_n(Q(u, P_m(Q(u_1, \ldots, u_k, v; P)); Q')))$$

which is convergent by using the sewing axiom and the fact that $\lambda \in \tilde{G}$. Moreover it is clear that this iterated sum is the expansion of an analytic function in two variables evaluated at a certain particular point. Thus the double sum must be absolutely convergent and consequently the right-hand side of (3.1) is absolutely convergent. ∎

For any $l \geq 0$, we define a linear map $\alpha_l : \tilde{G} \otimes X^{l+1} \otimes F_l^* \to V^*$ by

$$\begin{aligned}
&(\alpha_l(\lambda \otimes v_1 \otimes \ldots \otimes v_l \otimes v \otimes \mu))(u)\\
&\quad = \langle u \diamond_{\tilde{Q}} \lambda, e_l(v_1 \otimes \ldots \otimes v_l \otimes v \otimes \mu)\rangle.
\end{aligned}$$

for $\lambda \in \tilde{G}$, $v_1, \ldots, v_l, v \in X$, $\mu \in F_l^*$ and $u \in V$.

Proposition 3.2 *The image of α_l is in $\tilde{G}$.*

Proof. For any $k \geq 0$, $\lambda \in \tilde{G}$, $u_1, \ldots, u_k, u \in V$, $P \in \Theta_k$, $v_1, \ldots, v_l, v \in X$ and $\mu \in F_l^*$,

$$\sum_{n\in\mathbb{Z}} (\alpha_l(\lambda \otimes v_1 \otimes \ldots \otimes v_l \otimes v \otimes \mu))(P_n(Q(u_1, \ldots, u_k, u; P)))$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \langle (P_n(Q(u_1,\dots,u_k,u;P)) \diamond_{\tilde{Q}} \lambda), e_l(v_1\otimes\cdots\otimes v_l\otimes v\otimes\mu)\rangle \\
&= \sum_{n\in\mathbb{Z}} \mu(g_l((P_n(Q(u_1,\dots,u_k,u;P)) \diamond_{\tilde{Q}} \lambda)\otimes v_1\otimes\cdots\otimes\otimes v_l\otimes v)) \\
&= \sum_{n\in\mathbb{Z}} \mu\Bigg(\sum_{m\in\mathbb{Z}} (P_n(Q(u_1,\dots,u_k,u;P)) \diamond_{\tilde{Q}} \lambda)(P_m(Q(v_1,\dots,v_k,v;\cdot)))\Bigg) \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} \mu(\lambda(P_p((\Psi_2(\tilde{Q}')) \\
&\qquad\qquad ((P_n(Q(u_1,\dots,u_k,u;P))\otimes P_m(Q(v_1,\dots,v_k,v;\cdot)))) \\
&= C\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} \mu(\lambda(P_p((\nu_2(\tilde{Q}')) \\
&\qquad\qquad ((P_n(Q(u_1,\dots,u_k,u;P))\otimes P_m(Q(v_1,\dots,v_k,v;\cdot)))).
\end{aligned}
\tag{3.2}
$$

We need only to show that the right hand side of (3.2) is absolutely convergent. The proof is similar to the proof in Proposition 3.1 above: We first show that one of the iterated sums in other orders is absolutely convergent and is convergent to an analytic function in Q. Then this function can be expanded as series and the series is triply absolutely convergent. In particular, the iterated sum in the right-hand side of (3.2) is absolutely convergent and is equal to this triple sum. ∎

By Proposition 3.2,

$$\sum_{n\in\mathbb{Z}} \alpha_l(\lambda\otimes v_1\otimes\ldots\otimes v_l\otimes v\otimes\mu)(P_n(Q(u_1,\dots,u_k,u;\tilde{Q})))$$

is absolutely convergent and equal to

$$g_k(\alpha_l(\lambda\otimes v_1\otimes\ldots\otimes v_l\otimes v\otimes\mu)\otimes u_1\otimes\cdots\otimes u_k\otimes u)\in F_k.$$

We define a linear map

$$\beta_{k,l}: F_k^*\otimes F_l^*\to F_{k+l+1}^*$$

by

$$
\begin{aligned}
&(\beta_{k,l}(\mu_1,\mu_2))(g_{k+l+1}(\lambda\otimes u_1\otimes\ldots\otimes u_{k+1}\otimes u\otimes v_1\otimes\cdots\otimes v_l\otimes v)) \\
&\quad = \mu_1(g_k(\alpha_l(\lambda\otimes v_1\otimes\ldots\otimes v_l\otimes v\otimes\mu_2)\otimes u_1\otimes\cdots\otimes u_k\otimes u))
\end{aligned}
$$

for $\lambda \in \tilde{G}$, $u_1, \ldots, u_k, u, v_1, \ldots, v_l, v \in V$, $\mu_1 \in F_k^*$ and $\mu_2 \in F_l^*$. In fact this formula only gives a linear map from $F_k^* \otimes F_l^*$ to the algebraic dual of F_{k+l+1}. The proof of the following result is completely analogous to Proposition 2.3 in [H4]:

Proposition 3.3 *The image of the map $\beta_{k,l}$ is indeed in F_{k+l+1}^* and the map $\beta_{k,l}$ is continuous.* ∎

Let
$$h_1 = e_k(u_1 \otimes \cdots \otimes u_k \otimes u \otimes \mu_1) \in G_k$$
and
$$h_2 = e_l(v_1 \otimes \cdots \otimes v_l \otimes v \otimes \mu_2) \in G_l$$
where $u_1, \ldots, u_k, u, v_1, \ldots, v_l, v \in X$, $\mu_1 \in F_k^*$ and $\mu_2 \in F_l^*$. We define

$$\begin{aligned}&(\overline{\Psi}_Y(\tilde{Q}))(h_1 \otimes h_2)\\ &\qquad = e_{k+l+1}(u_1 \otimes \cdots \otimes u_k \otimes u \otimes v_1 \otimes \cdots \otimes v_l \otimes v \otimes \beta_{k,l}(\mu_1, \mu_2)).\end{aligned}$$

Note that any element of G_k or G_l is a linear combination of elements of the form h_1 or h_2, respectively, given above, and that k and l are arbitrary. Thus we obtain a linear map

$$\overline{\Psi}_Y(\tilde{Q})\big|_{G \otimes G} : G \otimes G \to G.$$

The proof of the following result is completely analogous to the proof of Proposition 2.4 in [H4]:

Proposition 3.4 *The map $\overline{\Psi}_Y(\tilde{Q})\big|_{G \otimes G}$ is continuous.* ∎

Since G is dense in H, we can extend $\overline{\Psi}_Y(\tilde{Q})\big|_{G \otimes G}$ to a linear map $\overline{\Psi}_Y(\tilde{Q})$ from $H \widetilde{\otimes} H$ to H. The proof of the following theorem is completely analogous to the proof of Theorem2.5 in [H4]:

Theorem 3.5 *The map $\overline{\Psi}_Y(\tilde{Q})$ is a continuous extension of $\Psi_Y(\tilde{Q})$ to $H \widetilde{\otimes} H$. That is, $\overline{\Psi}_Y(\tilde{Q})$ is continuous and*

$$\overline{\Psi}_Y(\tilde{Q})\big|_{V \otimes V} = \Psi_Y(\tilde{Q}). \qquad ∎$$

4 Locally convex completions, operads and double loop spaces

In this section, we reformulate the result obtained in [H4] and in Sections 2 and 3 above using the language of operads.

First, the result in Section 2 of [H4] immediately gives the following:

Theorem 4.1 *Let V be a finitely-generated $\mathbb{Z}$-graded vertex algebra. Then the topological completion H of V constructed in [H4] has a structure of an algebra over the framed little disk operad such that for the unit disk with two embedded disks of radius r_1 and r_2 centered at 0 and z, respectively, the corresponding map from $H\widetilde{\otimes}H$ to H is the map $\overline{\nu}_Y([D(z, r_1, r_2)])$. (See [H4] for the notation $\overline{\nu}_Y$ and $[D(z, r_1, r_2)]$.)*

Proof. The framed little disk operad is generated by the unit disk with two embedded disks of radius r_1 and r_2 centered at 0 and z and the unit disk with the unit disk itself embedded and with the frames given by complex numbers a of absolute value equal to 1. So we need only define the maps corresponding to these elements of the operad. For the unit disk with two embedded disks of radius r_1 and r_2 centered at 0 and z, we define the associated map to be $\nu_Y([D(z, r_1, r_2)])$. For the unit disk with the unit disk itself embedded and with the frames given by complex numbers a of absolute value equal to 1, we define the associated map to be $a^{L(0)} : H \to H$. Then we get a structure of algebra on H over the framed little disk operad. ∎

Next, combining the results of [H3] and the results in Sections 2 and 3 above, we obtain the following result:

Theorem 4.2 *Let V be a finitely-generated vertex operator algebra. Then the topological completion H of V constructed above has a structure of an algebra over the operad $\tilde{K}^c_{\mathfrak{H}_1}$ or, equivalently, of* $\mathrm{Det}^{c/2}$. ∎

Corollary 4.3 *Let V be a finitely-generated $\mathbb{Z}$-graded vertex algebra or a finitely-generated vertex operator algebra. Then locally convex completion H of V constructed in Part I ([H4]) or in Section 1 above has a structure of a space over the little framed disk operad. In particular, it has a structure of a space over the little framed disk operad.*

Proof. Since we have a natural continuous map from $H \times H$ to $H \otimes H$, we see from Theorem 4.1 that when V is a finitely-generated $\mathbb{Z}$-graded vertex algebra, its locally convex completion constructed in Part I has a structure of a space over the little framed disk operad.

If V is a finitely-generated vertex operator algebra. Then note that the little framed disk operad can in fact be viewed as a suboperad of $K_{\mathfrak{H}_1}$. Also note that the sewing of the determinant lines over elements in the little framed disk operad is trivial (see Appendix D of [H3]). Thus H has a structure of an algebra over the framed little disk operad and consequently has a structure of a space over the little framed disk operad. ∎

A subspace of a Hausdorff space is said to be *compactly closed* if the intersection of the subspace with each compact subset of the Hausdorff space is closed. A Hausdorff space is said to be *compactly generated* if every compactly closed subspace is closed. See [St] (and [W] and [M2]) for the notion of compactly generated topological space and properties of these spaces. In [M1], May proved, among other things, the following recognition principle for double loop spaces:

Theorem 4.4 *If a compactly generated Hausdorff based topological space has a structure of a space over the little disk operad, then it has the weak homotopy of a double loop space.* ∎

From [St] (see also [W] and [M2]), we know that we can make a Hausdorff space into a compactly generated Hausdorff space by giving it a new topology in which a subspace is closed if and only if it is compactly closed in the original topology. Since this functor is usually denoted by k, here we call the space with the new compactly generated topology the *k-ification* of the original space. Note that in the category of compactly generated spaces, the product of spaces is defined to be the k-ification of the usual product (see [St], [W] and [M2]). The following lemma follows immediately from the properties of k-ifications of topological spaces:

Lemma 4.5 *If a Hausdorff based topological space is a space over the little disk operad (with the usual products of topological spaces), then the k-ification of the space has a natural structure of a space over the little disk operad (with the products of compactly generated spaces).* ∎

Combining Corollary 4.3 with Theorem 4.4 and Lemma 4.5, we obtain:

Theorem 4.6 *The k-ifications of the locally convex completions constructed in [H4] and in Section 1 above have weak homotopy types of double loop spaces.* ∎

5 Locally-grading restricted conformal vertex algebras and topological completions

The results in the present paper are true also for algebras which do not satisfy the (global) grading-restriction conditions. We first need the following:

Definition 5.1 A *conformal vertex algebra* is a $\mathbb{Z}$-graded vertex algebra equipped with a Virasoro element ω satisfying all the axioms for vertex operator algebras except the two grading-restriction axioms. A conformal vertex algebra is said to be *locally grading-restricted* if for any element of the conformal vertex algebra, the module $W = \coprod_{n\in\mathbb{Z}} W_{(n)}$ for the Virasoro algebra generated by this element satisfies the grading-restriction conditions, that is, $\dim W_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $W_{(n)} = 0$ for n sufficiently small.

Remark 5.2 In fact, it is not difficult to show that the condition $\dim W_{(n)} < \infty$ in the definition above can be derived as a consequence. Thus for concrete examples, one need only verify the lower-truncation condition $W_{(n)} = 0$ for n sufficiently small.

We have the following:

Theorem 5.3 *The constructions and results in [H4] and in Sections 1, 2, 3 and 4 above hold for finitely-generated locally-grading-restricted conformal vertex algebras.*

Proof. Note that the constructions and results in [H4] and in Sections 1, 2, 3 and 4 above need only the locally grading-restriction conditions: All the properties of vertex operator algebras used, for example, commutativity, associativity, rationality and the factorization of exponentials of infinite sums of Virasoro operators, still hold if the locally-grading-restriction conditions are satisfied. The details are left to the reader as an exercise. ∎

Remark 5.4 Theorem 5.3 has been used in [HZ].

6 A locally convex completion of a finitely-generated module and operads

We give the results for modules in this section. Since the constructions and proofs are all similar to the case of algebras, we shall only state the final results. All the constructions and proofs are left to the reader as exercises.

Theorem 6.1 *Let V be a finitely-generated vertex operator algebra of central charge c, H its locally convex topological completion constructed in Section 1 and W a finitely-generated V-module. Then constructions completely analogous to those in Sections 1, 2, 3 and 4 above give a locally convex topological completion H^W of W and a structure of a module for the algebra H over the operad $\tilde{K}^c_{\mathfrak{H}_1}$ (or equivalently of $\mathrm{Det}^{c/2}$) on H^W.* ∎

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